BANDLIMITED FIELD RECONSTRUCTION FROM SAMPLES OBTAINED ON A DISCRETE GRID WITH UNKNOWN RANDOM LOCATIONS

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ABSTRACT

Sampling spatial fields using sensors which are location unaware is an exciting topic. Due to symmetry and shift-invariance of bandlimited fields, it is known that uniformly distributed location-unaware sensors cannot infer the field.

This work studies asymmetric (nonuniform) distributions on location-unaware sensors that will enable bandlimited field inference. In this first exposition, to facilitate analysis, location-unaware sensors are restricted to a *discrete grid*. Oversampling is used to overcome the lack of location information. The samples obtained from location-unaware sensors are clustered together to infer the field using the probability distribution that governs sensor placement on the grid. Based on this clustering algorithm, the main result of this work is to find the *optimal* probability distribution on sensor locations that minimizes the detection error-probability of the underlying spatial field.

Index Terms— Signal reconstruction, signal sampling, wireless sensor networks

1. INTRODUCTION

Localization of sensors is a challenging task [1]. An alternate option to having expensive sensors or expensive localization algorithms is to work with sensors which are location unaware. Recently, bandlimited field estimation *without* any location information of the sensors in a distributed setup has been studied [2] and is an exciting topic. The key technique is to utilize multitude of such sensors (oversampling) and leverage the random distribution on their spatial locations. Due to symmetry and shift-invariance properties of bandlimited fields, it is known that uniformly distributed location-unaware sensors do not infer the field uniquely [2]. From now on, *location-unaware sensors will be simply termed as sensors*.

This work studies asymmetric (statistical) distributions on sensors, that may enable bandlimited field reconstruction. If the location of each sensor is random, then a bandlimited field operating on this randomness is observed. Such process is nonlinear and resulting inference problems are difficult. To overcome analytical intractability, in this first exposition on the topic, the sensors' location is restricted to a random point on an *equi-spaced discrete grid*.[1] Oversampling will be used to overcome location unawareness.

With oversampling, samples obtained from sensors can be clustered together to infer which sample belongs to which spatial location on the equi-spaced grid where the sensors are present. If p is probability with which a sensor falls at a given location, then $\approx np$ will be the number of samples obtained from there, as n becomes large. The success of this clustering scheme will depend on the probability distribution that governs sensor placement on the grid. By assigning locations to samples based on their expected frequency, the field can be *detected*. The *main result* of this work is to find the *optimal* probability distribution on sensor locations that minimizes the detection error-probability of the underlying spatial field.

Prior art: Estimation of bandlimited fields from samples taken at unknown but statistically distributed locations was studied by Kumar [2]. Reconstruction of discrete-time bandlimited fields from unknown sampling locations was studied by Marziliano and Vetterli [3] in a combinatorial setting. Estimation of bandlimited signals with random sampling locations has been studied by Nordio et al. [4], where the locations are obtained by a perturbation of the equi-spaced grid. In this work, design of probability distribution on sensor locations is addressed to minimize the detection error-probability.

Notation: Space will be denoted by t. Spatial field will be denotes by $g(t)$. Vectors are column-vectors. The probability operator will be denoted by $\mathbb{P}$. Indicator function of a set A will be denoted by $\mathbb{1}(x \in A)$. Independent and identically distributed will be termed as i.i.d. Finally, $j = \sqrt{-1}$.

Organization: Sampling model and existing results necessary for analysis are presented in Section 2. In Section 3, the minimization of field detection error-probability is addressed. Finally, conclusions are presented in Section 4.

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[1]This may arise in scenarios where location information is masked to preserve the identity of the sensors, or to reduce the amount of transmitted-data.

2. SAMPLING MODEL AND REVIEW

Sampling model used and review of related theoretical results is discussed in this section. Field model appears first.

2.1. Spatial field model

The spatial field $g(t)$ is assumed to be periodic, real-valued and bandlimited. Without loss of generality (WLOG), the period of $g(t)$ is fixed to 1. Then, the Fourier series of $g(t)$ is

$$g(t) = \sum_{k=-b}^{b} a[k] \exp(j2\pi kt) \quad (1)$$

where $a[k]$ are the Fourier series coefficients of $g(t)$ and b is a *known* bandwidth parameter. For simplicity of notation, define $s_b := 1/(2b+1)$ as a spacing parameter and $\phi_k := \exp(j2\pi k s_b), -b \leq k \leq b$. Let Φ_b be defined as

$$\Phi_b = \begin{bmatrix} 1 & \dots & 1 \\ \phi_{-b} & \dots & \phi_b \\ \vdots & & \vdots \\ (\phi_{-b})^{2b} & \dots & (\phi_b)^{2b} \end{bmatrix}.$$

The columns of Φ_b are orthogonal and a sampling theorem ensures that [5, 4]:

$$\vec{a} = (\Phi_b)^{-1}\vec{g} = \frac{1}{(2b+1)}\Phi_b^{\dagger}\vec{g}, \quad (2)$$

where $\vec{a} = (a[-b], a[-b+1], \dots, a[b])^T$, where $\Phi_b^{\dagger}$ is the conjugate transpose of Φ_b, and $\vec{g} = (g(0), g(s_b), \dots, g(2bs_b))^T$. From (2), $\vec{a}$ and $g(t)$ can be obtained using the samples in $\vec{g}$.

It will be assumed that $g(is_b)$ are *distinct* for different values of i. This feature will be useful during clustering.[2]

2.2. Sensor deployment model

A discrete-valued non-uniform distribution is considered for bandlimited field inference. It will be assumed that a sensor is at location T such that $T = is_b$ with probability p_i where $i = 0, 1, \dots, 2b$ and $\sum_{i=0}^{2b} p_i = 1$. Correspondingly,

$$g(T) = g(is_b) \text{ with probability } p_i, \quad i = 0, 1, \dots, 2b \quad (3)$$

In our model, the sensor falls at $is_b, 0 \leq i \leq 2b$ but its location, that is the index i, is *not known*. The parameter $\vec{p} := p_0, p_1, \dots, p_{2b}$ will be treated as a *design choice* to optimize any performance criterion (see Section 3). It will be assumed that elements of $\vec{p}$ are distinct (to break symmetry in the distribution of sensor-locations). WLOG, assume that

$$p_0 < p_1 < \dots < p_{2b}. \quad (4)$$

It will be assumed that i.i.d. samples $g(T_1), g(T_2), \dots, g(T_n)$ are available for the detection of spatial field, where n corresponds to oversampling.[3]

2.3. Useful mathematical results

To analyze the detection error-probability, large deviation analysis setup will be used. Sanov's theorem, which addresses the asymptotic likelihood properties with respect to an incorrect probability model, will be used [6, Chap 11.4]. Let $X_1, \dots, X_n$ be i.i.d. random variables with discrete distribution $\vec{p}$. Then, the observed distribution of $X_1, \dots, X_n$ lies in the closed set E with the following probability

$$\lim_{n\to\infty} \frac{1}{n} \log_2 [\mathbb{P}(X_1^n \in E)] = -D(\vec{q}* \parallel \vec{p}) \quad (5)$$

where $\vec{q}* = \arg\min_{\vec{q}\in E} D(\vec{q} \parallel \vec{p})$ is the distribution in E that is the closest to $\vec{p}$ in the Kullback Leibler divergence or relative entropy terms. The quantity $D(\vec{q}* \parallel \vec{p})$ will be termed as the *error-exponent* in this work.

The following inequalities will be used for optimization

$$\text{AM-GM: } \frac{x+y}{2} \geq \sqrt{xy}, \quad x, y \geq 0 \quad (6)$$

$$\text{Log-sum: } \sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq a \log \frac{a}{b}, \quad a_i, b_i > 0 \quad (7)$$

where $a = \sum_{i=1}^{n} a_i$ and $b = \sum_{i=1}^{n} b_i$.

3. FIELD DETECTION AND ITS PERFORMANCE

Field detection and the choice of $\vec{p}$ is discussed in this section.

3.1. The field detection algorithm

Based on the readings $g(T_1), g(T_2), \dots, g(T_n)$, the field $g(t)$ has to be detected. From (4) and Section 2.1, $\{g(is_b), p_i\}$ pairs are distinct in both the elements. Each sensor records $g(is_b)$ with probability p_i. The following *clustering algorithm* will be used to ascertain the field samples $g(is_b)$, which specify the entire field $g(t)$ (see (2)):

1. The readings $Y_1 := g(T_1), \dots, Y_n := g(T_n)$, with T_i unknown and in the set $\{0, s_b, \dots, 2bs_b\}$, are collected.
2. The values $Y_1, Y_2, \dots, Y_n$ are clustered into (*value*, *type*) pairs. Equal values (*value*) in $Y_1, Y_2, \dots, Y_n$ are collected together and the number of equal values (*type*) is recorded.

[2] If $\vec{a}$ is the realization of a continuous random distribution, then this condition will hold almost surely. A violation of this condition implies that $\sum_{k=-b}^{b} a[k](\exp(j2\pi kms_b) - \exp(j2\pi kns_b)) = 0$. That is, a linear combination of $\vec{a}$–a continuous random variable–is zero with probability one.

[3] It is desirable to address the setup where each sensor's location T is realized from an asymmetric continuous distribution supported in $[0, 1]$. This problem is nonlinear and analytically difficult.

3. Empirical probabilities *type*$/n$ for each *value* are calculated. For large n, the empirical probability *type*$/n$ of each *value* will be near the correct p_i in $\vec{p}$.

4. The *value* with smallest empirical probability is assigned to $g(0)$, the *value* with next smallest empirical probability is assigned to $g(s_b)$, and so on till $g(2bs_b)$.

Example 3.1 *Consider a signal $g_1(t)$ with bandwidth parameter $b = 1$, and $s_b = \frac{1}{2b+1} = \frac{1}{3}$. The field values are known to be $g_1(0) = 1.06, g_1(1/3) = 1.80, g_1(2/3) = 0.14$.*

The field is sampled using $n = 10$ randomly realized values of sensor's location in the set $\{0, 1/3, 2/3\}$. The 10 observed samples were $1.80, 0.14, 0.14, 1.06, 1.80, 0.14, 1.80, 1.06, 0.14, 0.14$. The (value, type) pairs are $(1.06, 2)$, $(1.80, 3)$, and $(0.14, 5)$. The above algorithm concludes that $g_1(0) = 1.06$, $g_1(1/3) = 1.80$, $g_1(2/3) = 0.14$, and is correct.

The field is again sampled using $n = 10$ randomly realized values of sensor's location. This time, the 10 observed samples were $1.06, 0.14, 0.14, 1.06, 1.80, 0.14, 1.80, 1.06, 0.14, 0.14$. The (value, type) pairs are $(1.06, 3)$, $(1.80, 2)$, and $(0.14, 5)$. The above algorithm concludes that $g_1(0) = 1.80$, $g_1(1/3) = 1.06$, $g_1(2/3) = 0.14$, and is incorrect.

For further discussions, define $N_i := \sum_{j=1}^{n} \mathbb{1}\left[Y_j = g(is_b)\right]$ as the type of $g(is_b)$ in n field observations. Then, in the above algorithm as $n \to \infty$, it is expected that

$$0 < N_0 < N_1 < \ldots < N_{2b}. \tag{8}$$

If the above event is violated, it results in erroneous field detection. The probability of correct detection in (8) will be maximized by choosing the sensor deployment distribution $\vec{p}$.

3.2. Detection error-probability minimization

The spatial field is detected correctly when the condition in (8) is satisfied. Let P_e be the detection error-probability. The error-exponent (as the number of samples n gets large) in the detection error-probability will be maximized. Note that,

$$\begin{aligned} P_e &= \mathbb{P}\Big[(0 < N_0 < N_1 < ... < N_{2b})^c\Big] \qquad (9) \\ &= \mathbb{P}\Big[\{N_0 = 0\} \cup \{N_0 \geq N_1\} \cup \ldots \cup \{N_{2b-1} \geq N_{2b}\}\Big] \end{aligned}$$

By applying the union-bound and the subset-inequality ($A \subseteq B$ implies $\mathbb{P}(A) \leq \mathbb{P}(B)$) in the above equation [7], we get

$$\begin{aligned} P_e \leq (2b+1) \max\Big\{&\mathbb{P}(N_0 = 0), \mathbb{P}(N_0 \geq N_1), \ldots, \\ &\mathbb{P}(N_{2b-1} \geq N_{2b})\Big\} \qquad (10) \end{aligned}$$

$$\begin{aligned} \text{and } P_e \geq \max\Big\{&\mathbb{P}(N_0 = 0), \mathbb{P}(N_0 \geq N_1), \ldots, \\ &\mathbb{P}(N_{2b-1} \geq N_{2b})\Big\}. \qquad (11) \end{aligned}$$

From the above equations, the error-exponent in P_e is maximized if the error exponent of $\max\Big\{\mathbb{P}(N_0 = 0), \mathbb{P}(N_0 \geq N_1), \ldots, \mathbb{P}(N_{2b-1} \geq N_{2b})\Big\}$ is maximized. The constant factor $(2b + 1)$ in (10) does not contribute to the error-exponent. The error-exponent maximization is addressed next.

A sensor falls at location 0 with probability p_0. With n randomly deployed sensors,

$$\mathbb{P}[N_0 = 0] = (1 - p_0)^n. \tag{12}$$

To compute $\mathbb{P}[N_0 \geq N_1]$ and other similar events, Sanov's theorem will be used (see (5)). An empirical distribution $\vec{q}$ will be found such that $D(\vec{q} \parallel \vec{p})$ is minimum, which results in the error-exponent via Sanov's theorem (see (5)). The empirical distribution is $\vec{q} = \left[\frac{N_0}{n}, \frac{N_1}{n}, \ldots, \frac{N_{2b}}{n}\right]$ and, from Sanov's theorem, the function to be minimized is

$$\begin{aligned} D(\vec{q} \parallel \vec{p}) &= \sum_{i=0}^{2b} \frac{N_i}{n} \log_2 \frac{N_i}{np_i} \\ \text{subject to } &\sum_{i=0}^{2b} \frac{N_i}{n} = 1 \text{ and } N_1 \leq N_0. \end{aligned} \tag{13}$$

The corresponding Lagrangian is

$$L = \sum_{i=0}^{2b} \frac{N_i}{n} \log_2 \frac{N_i}{np_i} + \lambda \left(\sum_{i=0}^{2b} N_i - n \right) + \mu(N_1 - N_0)$$

At the minima of $D(\vec{q} \parallel \vec{p})$ in (13),

$$\frac{\partial L}{\partial N_i} = 0 \quad \text{for } 0 \leq i \leq 2b \tag{14}$$

The solutions of above equation are

$$N_0 = \frac{np_0}{e} 2^{-n(\lambda - \mu)}, N_1 = \frac{np_1}{e} 2^{-n(\lambda + \mu)}, \tag{15}$$

and,

$$N_i = \frac{np_i}{e} 2^{-n\lambda} \text{ for } i \geq 2. \tag{16}$$

The values of μ and λ can be found by KKT conditions [8], but by using the log-sum and AM-GM inequalities in (7) and (6) μ can be found directly as follows. Observe that μ is only associated with N_0 and N_1. The terms corresponding to N_0 and N_1 in (13) is lower-bounded by

$$\begin{aligned} \frac{N_0}{n} \log_2 \frac{N_0}{np_0} + \frac{N_1}{n} \log_2 \frac{N_1}{np_1} &\geq \frac{N_0 + N_1}{n} \log_2 \frac{N_0 + N_1}{n(p_0 + p_1)} \\ &\geq \frac{2\sqrt{N_0 N_1}}{n} \log_2 \frac{2\sqrt{N_0 N_1}}{n(p_0 + p_1)}. \end{aligned}$$

In the above equation, the minimum value requires that $N_0 = N_1$. This results in

$$\mu = \frac{1}{2n} \log_2 \frac{p_1}{p_0}, \text{ and } N_0 = N_1 = \frac{n}{e} 2^{-n\lambda} \sqrt{p_0 p_1} \tag{17}$$

In (15), the product $N_0 N_1$ does not depend on μ. So the minimum value of first two terms in $D(\vec{q} \parallel \vec{p})$ is attained only when $N_0 = N_1 = \frac{n}{e} 2^{-n\lambda}\sqrt{p_0 p_1}$.

For finding λ, note that $\sum_{i=0}^{2b} N_i = n$. Using N_0, N_1 from (17) and N_i from (16) results in

$$\lambda = -\frac{1}{n}\log_2\left(\frac{e}{1-(\sqrt{p_1}-\sqrt{p_0})^2}\right) \tag{18}$$

This value of λ gives

$$N_i = \frac{np_i}{1-(\sqrt{p_1}-\sqrt{p_0})^2}$$
$$\text{and } N_0 = N_1 = \frac{n\sqrt{p_0 p_1}}{1-(\sqrt{p_1}-\sqrt{p_0})^2}.$$

Substitution of $N_0, N_1, \ldots, N_{2b}$ from the above equation in (13) results in the desired minimum value of $D(\vec{q}* \parallel \vec{p})$,

$$D(\vec{q}* \parallel \vec{p}) = \log_2 \frac{1}{1-(\sqrt{p_1}-\sqrt{p_0})^2} \tag{19}$$

For $N_i \geq N_{i+1}$, the optimization constraint $N_0 \geq N_1$ will get replaced by $N_i \geq N_{i+1}$ in (13). The analysis is identical and the result is

$$D(\vec{q}* \parallel \vec{p}) = \log_2 \frac{1}{1-(\sqrt{p_{i+1}}-\sqrt{p_i})^2}. \tag{20}$$

Let $d_0 = \sqrt{p_0}$ and $d_i = \sqrt{p_i} - \sqrt{p_{i-1}}$, $1 \leq i \leq 2b$ and let $d_{\min} = \min\{d_0, d_1, \ldots, d_{2b}\}$. Then $d_{\min}$ will determine the value of the largest term in $\max\Big\{\mathbb{P}(N_0 = 0), \mathbb{P}(N_0 \geq N_1), \ldots, \mathbb{P}(N_{2b-1} \geq N_{2b})\Big\}$. This is by Sanov's theorem which asserts that $\mathbb{P}(N_i \geq N_{i+1}) \propto 2^{-nD(\vec{q}*\|\vec{p})}$. Consequently, the value of $d_{\min}$ has to be maximized.

For maximizing $d_{\min}$, note that

$$(2b+1)d_{\min} \leq \sum_{i=0}^{2b} d_i = \sqrt{p_{2b}}. \tag{21}$$

To satisfy equality in (21),

$$\sqrt{p_0} = \frac{\sqrt{p_{2b}}}{2b+1} \text{ and } \sqrt{p_{i+1}} = \sqrt{p_i} + \frac{\sqrt{p_{2b}}}{2b+1}. \tag{22}$$

This relationship, along with $p_0 + \ldots + p_{2b} = 1$, results in

$$p_i = \frac{3(i+1)^2}{(b+1)(2b+1)(4b+3)} \text{ for } 0 \leq i \leq 2b. \tag{23}$$

This law on $\vec{p}$ ensures that the field detection error probability in (9) is *minimized*, and is the *main result* of this work.

Using MATLAB, the detection error-probability was compared for different laws on $\vec{p}$. The field bandwidth was kept as $b = 4$ and its Fourier series coefficients were picked by a uniform random number generator: $a[0] = 1$, $a[1] = 0.9134 - j0.5469$, $a[2] = 0.1270 - j0.2785$, $a[3] = 0.9058 - j0.0975$, and $a[4] = 0.8147 - j0.6324$ were used. For real valued fields $a[-k] = \bar{a}[k]$ by conjugate symmetry. The number of randomly collected samples was varied between 100 to 10000. The empirical probability of field detection error, when calculated using 10000 Monte-Carlo trials, is plotted in Fig. 1. The log-log plot reveals the error exponent. Four different methods to select $\vec{p}$ were used for comparison. The selections include: (i) the optimal distribution in (23), (ii) a linear distribution $\vec{p} = [\alpha, 2\alpha, \ldots, (2b+1)\alpha]$, (iii) a cubic distribution $\vec{p} = [\alpha, 8\alpha, \ldots, (2b+1)^3\alpha]$, and (iv) ordered uniformly distributed random variable realizations based distribution $\vec{p} = \alpha[U(1), U(2), \ldots, U(2b+1)]$. In all these cases, α was selected to ensure $\sum_{i=0}^{2b} p_i = 1$. Each plot ends when the empirical detection error probability becomes zero. From the plots, the distribution discovered in (23) has fastest decay, and it results in smallest detection error-probability.

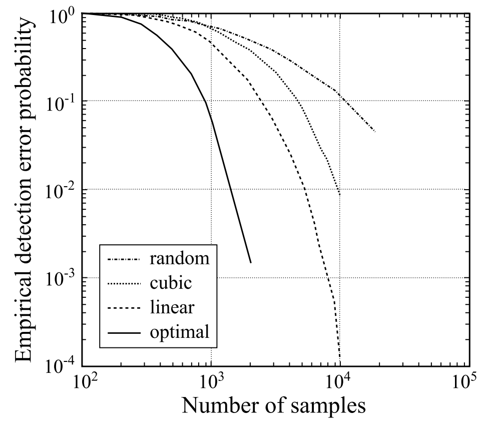

Fig. 1. Detection error-probabilities for different laws on $\vec{p}$ are compared. The four laws used include the optimal $\vec{p}$ in (23), a linear law, a cubic law, and a randomly generated $\vec{p}$. As expected, the law in (23) is the best in performance.

4. CONCLUSIONS

The detection of bandlimited fields using location-unaware sensors was addressed in this work. The sensor locations were restricted to an equi-spaced discrete grid. Using an algorithm, which clusters distinct field values and records their types, field detection can be performed. It was shown that the detection error-probability decreases exponentially fast in the number of sensors deployed. The optimal distribution for maximizing the error-probability exponent was derived.

In the presence of measurement-noise, the samples will not belong to a finite set and necessitate the use of more sophisticated clustering algorithms. This setup and the setup where sensors are located with an arbitrary continuous distribution are left for future work.

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