FAST AND EFFICIENT DISTRIBUTED MATRIX-VECTOR MULTIPLICATION USING RATELESS FOUNTAIN CODES

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ABSTRACT

We propose a *rateless fountain coding* strategy to alleviate the problem of straggling nodes – computing nodes that unpredictably slowdown or fail – in distributed matrix-vector multiplication. Our algorithm generates linear combinations of the m rows of the matrix, and assigns them to different worker nodes, which then perform row-vector products with the encoded rows. The original matrix-vector product can be decoded as soon as slightly more than m row-vector products are collectively completed by the nodes. This strategy enables fast nodes to steal work from slow nodes, without requiring the knowledge of node speeds. Compared to recently proposed fixed-rate erasure coding strategies which ignore partial work done by straggling nodes, rateless codes have a significantly lower overall delay, and a smaller computational overhead.

***Index Terms*—** Distributed Computing, Straggler Mitigation, Erasure Codes

1. INTRODUCTION

Matrix-vector multiplications form the core of a plethora of scientific computing and machine learning applications that include solving partial differential equations [1], forward and back propagation in neural networks [2], computing the PageRank of graphs [3] etc. In this age of Big Data, most of these applications involve multiplying extremely large matrices and vectors and the computations cannot be performed efficiently on a single machine and are instead distributed across multiple computation nodes. Individual nodes (the *workers*) perform their respective tasks in parallel while a central node (the *master*) aggregates the output of all the workers to complete the computation. Unfortunately, such approaches are often bottlenecked by a few slow or unresponsive workers, called stragglers [4], that delay the entire computation as the master needs to wait for all workers to complete their assigned tasks.

In the past, the problem of stragglers has been addressed by replicating tasks at individual workers [5, 6, 7], and waiting for any one copy to finish. The observation that replication is a special case of the more general erasure coding framework wherein stragglers can be modeled as *erasures* has led to the employment of Maximum Distance Separable (MDS) codes [8, 9, 10, 11] to speed up the computation of matrix vector products in a distributed setting. In this framework codes are employed to add redundancy so that only a subset of nodes are required to complete the tasks assigned to them.

In this work we use rateless fountain codes [12, 13] for distributed multiplication of a $m \times n$ matrix $\mathbf{A}$ with a $n \times 1$ vector $\mathbf{x}$. The rateless coded matrix-vector multiplication algorithm generates coded linear combinations of the m rows of matrix $\mathbf{A}$ and distributes them across p worker nodes. Each node also receives a copy of $\mathbf{x}$. The master node needs to wait for any $m_d = m(1+\epsilon)$ row-vector products to be completed across *all* the nodes, where ϵ is a small overhead ($\epsilon \to 0$ as $m \to \infty$). Using rateless codes has the following key benefits.

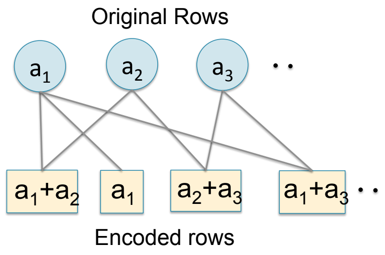
Low Latency via Seamless Load-Balancing. A key drawback of replication or fixed-rate coding strategies is that they rely on using the results of a fast subset of worker nodes, and completely ignore partial computations performed by slow or straggling nodes. On the other hand our rateless coding strategy achieves near *ideal* load balancing by utilizing the computations performed by *all* nodes while ensuring that faster nodes complete more tasks than slower nodes. This also provides robustness to node failures since available nodes can make up for computations lost due to failed nodes.

Negligible Redundant Computation. Rateless Coding performs $m_d = m(1+\epsilon)$ row-vector product computations on average, where, the overhead ϵ goes to zero as the number of matrix rows m increases. In the case of MDS coding or replication, if there is no straggling, the worker nodes collectively perform a large amount of redundant computation.

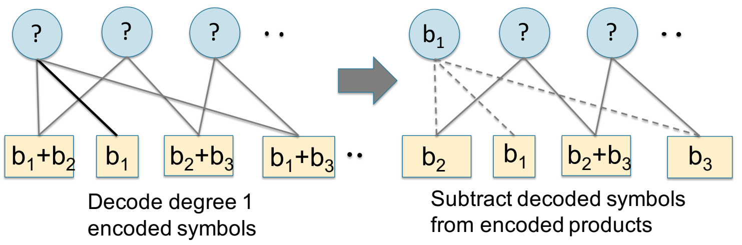
Low Decoding Complexity. Another benefit of using rateless codes is the extremely fast decoding of $\mathcal{O}(m \ln m)$ which allows our approach to scale efficiently even for very large m.

While the use of LT codes for matrix-vector multiplication has been recently proposed in [14, 15], these works do not utilize partial work done by stragglers, which is the key novel contribution of our work. Due to this our approach achieves the aforementioned benefits of near optimal straggler tolerance, negligible overhead of redundant computation, as well as robustness to node failures.

[*]Malhar Chaudhari performed the work while at CMU



(a) Encoding Graph


?
?
?
b1+b2
b1
b2+b3
b1+b3
Decode degree 1 encoded symbols
b1
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?
b2
b1
b2+b3
b3
Subtract decoded symbols from encoded products

(b) Decoding Graph

Fig. 1: (a) Bipartite graph representation of the encoding of the rows $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m$ of matrix $\mathbf{A}$. Each encoded row is the sum of d rows of $\mathbf{A}$ chosen uniformly at random, where d is drawn from the Robust Soliton degree distribution [12]. (b) In each step of the iterative decoding process, a single degree one encoded symbol is decoded directly, and is subtracted from all sums in which it participates.

2. PROBLEM FORMULATION

Consider the problem of multiplying a $m \times n$ matrix $\mathbf{A}$ with a $n \times 1$ vector $\mathbf{x}$ using p worker nodes and a master node. The worker nodes can only communicate with the master, and cannot directly communicate with other workers. The goal is to compute the result $\mathbf{b} = \mathbf{Ax}$ in a distributed fashion and mitigate the effect of unpredictable node slowdown or straggling. The rows of matrix $\mathbf{A}$ are encoded using an erasure code to give the $m_e \times n$ encoded matrix $\mathbf{A_e}$, where $m_e = \alpha m \geq m$. Matrix $\mathbf{A_e}$ is split along its rows to give p submatrices $\mathbf{A}_{\mathbf{e},1}, \ldots, \mathbf{A}_{\mathbf{e},p}$ of equal size such that worker i stores submatrix $\mathbf{A}_{\mathbf{e},i}$. To compute the matrix-vector product $\mathbf{b} = \mathbf{Ax}$, the vector $\mathbf{x}$ is communicated to the workers such that Worker i is tasked with computing the product $\mathbf{A}_{\mathbf{e},i}\mathbf{x}$.

To complete the assigned task, each worker needs to compute a sequence of row vector products of the form $\mathbf{a}_{\mathbf{e},j}\mathbf{x}$ where $\mathbf{a}_{\mathbf{e},j}$ is the j^{th} row of $\mathbf{A_e}$. The time taken by a worker node to finish computing one or more row-vector products may be random due to variability in the node speed or the amount of computation assigned to it. The master node aggregates the computations of all, or a subset of, the workers into the vector $\mathbf{b_e}$, which is then decoded to give the final result $\mathbf{b} = \mathbf{Ax}$. If $\mathbf{b_e}$ is not decodable, the master waits until more row-vector products are completed by the workers.

2.1. Performance Criteria

We use the following metrics to compare different distributed matrix-vector multiplication schemes via theoretical analysis and associated simulations (Section 4).

Definition 1 (Latency (T))**.** *The latency T is the time required by the system to complete enough number of computations so that $\mathbf{b} = \mathbf{Ax}$ can be successfully decoded from the worker computations aggregated in $\mathbf{b_e}$.*

Definition 2 (Computations (C))**.** *The number of computations C is defined as the total number of row-vector products $\mathbf{a}_{\mathbf{e},j}\mathbf{x}$ performed collectively by worker nodes until the vector $\mathbf{b} = \mathbf{Ax}$ is decoded.*

For any strategy, we have $C \geq m$ where m is the number of rows of $\mathbf{A}$ or the number of elements in $\mathbf{b}$.

2.2. Benchmarks for Comparison

We compare the performance of the proposed rateless coded strategy with two benchmarks: the replication, and the MDS-coded strategies.

The $r-$Replication Strategy. In this approach $\mathbf{A}$ is split along its rows into p/r submatrices $\mathbf{A}_1, \ldots, \mathbf{A}_{p/r}$, with rm/p rows each (assume that p/r divides m). Each submatrix is multiplied with $\mathbf{x}$ in parallel on r distinct worker nodes. The master collects the results from only the fastest of the r nodes that have been assigned the task of computing the product $\mathbf{A}_i\mathbf{x}$, for all i and discards the rest. The computed products are aggregated into the $m \times 1$ vector $\mathbf{b}$. Setting $r = 1$ corresponds to the naive or *uncoded* strategy where each submatrix-vector product is computed at a single worker node. Increasing the number of replicas provides greater straggler tolerance at the cost of more redundant computations. Systems like Mapreduce [16] and Spark [17] often use $r = 2$ i.e. each computation is assigned to 2 different worker nodes for added reliability and straggler tolerance.

The (p, k) MDS-Coded Strategy. The MDS-Coded appraoch for straggler mitigation [11, 18] involves pre-multiplying $\mathbf{A}$ at the master with a suitable encoding matrix $\mathbf{F}$ denoting the MDS code. Encoding $\mathbf{A}$ using a (p, k) MDS code involves splitting it along its rows into k matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$, each having m/k rows. The MDS code adds $p - k$ redundant matrices $\mathbf{A}_{k+1}, \ldots, \mathbf{A}_p$ which are independent linear combinations of the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$. Worker i computes the product of matrix $\mathbf{A}_i$ with vector $\mathbf{x}$. The master can recover the product $\mathbf{b} = \mathbf{Ax}$ if any k workers complete their task. Thus the system is robust to $p - k$ stragglers. However this strategy adds a significant computation overhead. When none of the nodes are slow, the system performs mp/k row-vector products, whereas in the uncoded case it only performs m row-vector products.

3. PROPOSED RATELESS CODING STRATEGY

We now propose our rateless coding strategy which uses LT codes [12] to mitigate the effect of stragglers in computing the matrix-vector product $\mathbf{b} = \mathbf{A}\mathbf{x}$ in the master-worker framework described in Section 2.

Luby Transform (LT) codes proposed in [12] are a class of erasure codes that can be used to generate a limitless number of encoded symbols from a finite set of source symbols. We apply LT codes to matrix-vector multiplication by treating the m rows of the matrix $\mathbf{A}$ as source symbols. Each encoded symbol is the sum of d source symbols chosen uniformly at random from the matrix rows. Thus if $\mathcal{S}_d \subseteq \{1, 2, \dots m\}$ is the set of d row indices, the corresponding encoded row is $\mathbf{a_e} = \sum_{i \in \mathcal{S}_d} \mathbf{a}_i$. The number of original rows in each encoded row, or the degree d, is chosen according to the Robust Soliton degree distribution the details of which are described in Appendix A and in Luby's original work [12].

The $m \times n$ matrix $\mathbf{A}$ is encoded to generate an $m_e \times n$ encoded matrix $\mathbf{A_e}$ where $m_e = \alpha m$. Once the αm rows of the encoded matrix $\mathbf{A_e}$ are generated, they are distributed equally among the p worker nodes. To multiply $\mathbf{A}$ with a vector $\mathbf{x}$, the master sends $\mathbf{x}$ to the workers. Each worker multiplies $\mathbf{x}$ with each row of $\mathbf{A_e}$ stored in its memory and returns the product (a scalar) to the master. The master collects row-vector products of the form $\mathbf{a}_{\mathbf{e},j}\mathbf{x}$ (elements of $\mathbf{b_e}$) from workers until it has enough elements to be able to recover $\mathbf{b}$.

To decode the desired matrix vector product $\mathbf{b} = \mathbf{A}\mathbf{x}$ from a subset of M' symbols of $\mathbf{b_e}$ we use the the iterative peeling decoder described in [12, 13]. If $\mathbf{b} = [b_1, b_2, \dots b_m]$, the decoder may receive symbols $b_1 + b_2 + b_3$, $b_2 + b_4$, b_3, b_4, and so on. Decoding is performed in an iterative fashion. In each iteration, the decoder finds a degree one encoded symbol, covers the corresponding source symbol, and subtracts the symbol from all other encoded symbols connected to that source symbols (see Figure 1b). Once the master receives M' elements of $\mathbf{b_e}$ (M' is the number of symbols required for successful decoding), it sends a *done* signal to all workers nodes to stop their local computation.

Since the encoding uses a random bipartite graph, the number of symbols required to decode the m source symbols successfully is a random variable M'. For the Robust Soliton distribution, [12] gives a high probability bound on M'.

Lemma 1 (Theorems 12 and 17 in [12])**.** *The original set of m source symbols can be recovered from a set of any $M' = m + \mathcal{O}(\sqrt{m}\ln^2(m/\delta))$ with probability at least $1 - \delta$.*

In our analysis we denote $m_d = \mathbb{E}[M']$. Using the theoretical guarantees for LT codes (see Appendix A) we can show that $m_d = m(1+\epsilon)$, where $\epsilon \to 0$ as $m \to \infty$. In practice one can choose a small enough value of δ and compute the corresponding value of M' such that decoding is susccessful with a high probability.

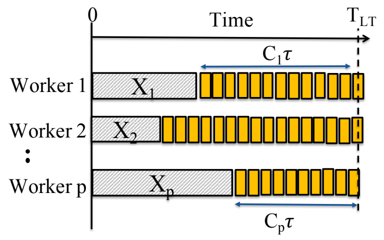


Fig. 2: Worker i has a random exponential initial delay X_i, after which it completes row-vector product tasks (denoted by the small rectangles), taking time τ per task. The latency T_{LT} is the time to complete M' tasks in total.

4. THEORETICAL ANALYSIS

Our main theoretical results involve comparing the proposed rateless coding strategy with the MDS and replication coding strategies in terms of latency and computations. The proofs of the theoretical results presented here are contained in Appendix B. We assume that worker i performs B_i computations in time Y_i where

$$Y_i = X_i + \tau B_i, \quad \text{for all } i = 1, \dots, p \tag{1}$$

where X_i is a random variable that includes the network latency, initial setup time, and other random components, and τ is a constant shift which is the time taken by any worker to perform a single computation (row-vector multiplication). When X_i is exponentially distributed with rate μ, the time taken by worker i to perform b computations is distributed as $\Pr(Y_i \le t) = 1 - \exp(-\mu(t - \tau b))$. While this follows the shifted exponential delay model used in [**?**, 9], the key difference is that the shift is parameterized by the number of computations at each worker. We believe this is a more realistic model as it captures the effect of increasing the amount of computations on the delay – if a worker is assigned more computations, there is larger delay. Figure 2 illustrates the latency of the LT coded strategy, T_{LT} under this delay model.

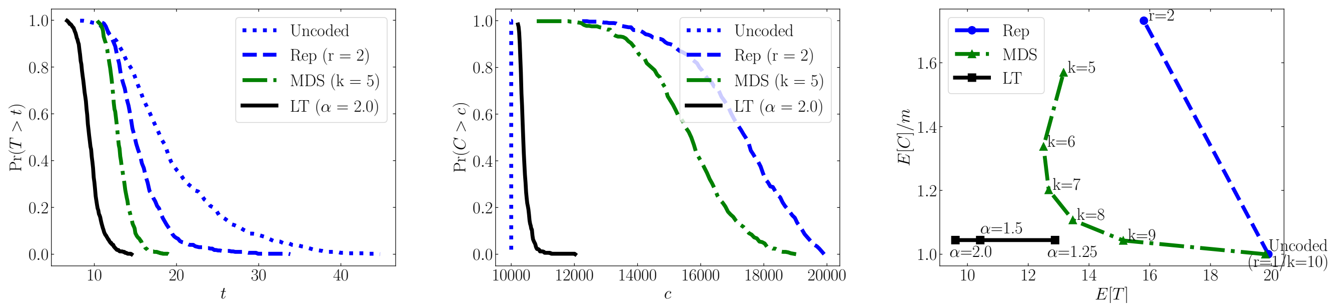
Theorem 1 (Latency of the Rateless Coded Strategy)**.** *For large m_e i.e. $\alpha = m_e/m \to \infty$, the expected latency for the LT-coded case with p workers and $X_i \sim \exp(\mu)$ for all workers $i = 1, \dots, p$, is bounded as.*

$$\mathbb{E}[T_{LT}] \le \frac{\tau m_d}{p} + \frac{1}{\mu} + \tau, \tag{2}$$

$$\mathbb{E}[T_{LT}] \ge \frac{\tau m_d}{p} + \frac{1}{p\mu}, \tag{3}$$

where $m_d = m(1+\epsilon)$ is the expected number of symbols necessary for successful decoding.

Remark 1. While we need $\alpha \to \infty$ for the above results to strictly hold, we observe empirically (Figure 3a) that LT codes exhibit superior latency performance over the benchmark strategies even for $\alpha = 2.0$.



(a) Latency Tail (b) Computation Tail (c) Avg. Computations v/s latency

Fig. 3: The tail probability of the latency is the highest for the replication schemes. MDS codes perform better in terms of latency but they perform a large number of redundant computations. The latency tail of LT codes is the minimum among all the schemes. Moreover the LT coded scheme performs significantly fewer redundant computations than MDS Codes or replication. All results were obtained from 500 Monte-Carlo Simulations with number of matrix rows, $m = 10000$, number of worker nodes $p = 10$ and delay model parameters $\mu = 0.2, \tau = 0.005$.

Theorem 2 (Latency of Replication and MDS Coding). *The expected latency for the $r-$Replication and the (p, k) MDS-coded strategies with $X_i \sim \exp(\mu)$ for all workers $i = 1, \dots, p$ is*

$$\mathbb{E}[T_{rep}] = \frac{\tau m r}{p} + \frac{1}{\mu} H_{p/r} \simeq \frac{\tau m r}{p} + \frac{1}{\mu} \log \frac{p}{r} \tag{4}$$

$$\mathbb{E}[T_{MDS}] = \frac{\tau m}{k} + \frac{1}{\mu}(H_p - H_{p-k}) \simeq \frac{\tau m}{k} + \frac{1}{\mu} \log \frac{p}{p-k}, \tag{5}$$

where $H_j = \sum_{v=1}^{j} 1/v$, the j^{th} Harmonic number.

Remark 2. Observe that in (4) and (5) above, adding redundancy (increasing r and reducing k respectively) leads to an increase in the first term (more computation at each node) and decrease in the second term (less delay due to stragglers). Thus, straggler mitigation comes at the cost of additional computation at the workers which might even lead to an increase in latency. This is seen in Figure 3c where the latency actually increases on adding redundancy for the MDS-coded case.

Remark 3. One of the main advantages of the rateless coded strategy is that the number of computations performed by all the workers asymptotically (as $m \to \infty$) approaches the minimum number of computations (m) required to recover a $m-$dimensional matrix-vector product (lemma 1). On the other hand, the following theorems show that the total computations performed by *all* workers (fast and slow) in the replication and MDS-coded schemes is much larger than m.

Theorem 3 (Tail of Computations for MDS Coding). *The tail of the number of computations of the MDS coded strategy, C_{MDS}, with p workers and $X_i \sim \exp(\mu)$ is bounded as*

$$\Pr(C_{MDS} \geq \frac{mp}{k} - C_0) \geq 1 - \exp\left(-\mu \theta_{MDS}\right) \tag{6}$$

$$\theta_{MDS} = \frac{\tau C_0}{(p-k)^2} - \frac{\tau}{p-k} \tag{7}$$

Theorem 4 (Tail of Computations for Replication). *The tail of the number of computations of the replication strategy, C_{rep}, with p workers and $X_i \sim \exp(\mu)$ is bounded as*

$$\Pr(C_{rep} \geq mr - C_0) \geq 1 - \sum_{i=0}^{p/r-1} \frac{1}{i!} \exp(-\mu\theta_{Rep})(\mu\theta_{Rep})^i \tag{8}$$

$$\theta_{rep} = \frac{\tau C_0}{(r-1)^2} - \frac{\tau p}{r(r-1)} \tag{9}$$

Remark 4. While the benefits of using partial work from all workers can be obtained by using any random linear code on the rows of $\mathbf{A}$, the key strength of LT codes is their low decoding complexity of $\mathcal{O}(m \ln m)$. Using an (m_e, m) MDS code over the rows of $\mathbf{A}$ then the decoding complexity would be $\mathcal{O}(m^3)$ which is unacceptable for large m.

5. CONCLUDING REMARKS

Due to the massive size of matrices arising in modern data-driven applications, computations such as matrix-vector multiplication need to be parallelized across multiple nodes. In this paper we propose an erasure coding strategy based on *rateless fountain codes* to overcome bottlenecks caused by slow or straggling nodes. For a matrix with m rows, our strategy requires the nodes to collectively finish slightly more than m row-vector products, and thus can seamlessly adapt to varying node speeds and achieve near-perfect load balancing. Moreover, it has a small overhead of redundant computations (asymptotically zero), and low decoding complexity. Thus, it strikes a better latency-computation trade-off than the uncoded and fixed-rate erasure coding strategies. The ideas of this paper can potentially be extended to any random linear network code including other rateless codes such as Raptor Codes [19] and other low-complexity random linear codes [20].

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A. PROPERTIES OF LT CODES

The number of original rows in each encoded row, or the degree d, is chosen according to the Robust Soliton degree distribution wherein the probability of choosing $d = i$ is proportional to

$$\rho(d) = \begin{cases} \frac{R}{dm} + \frac{1}{m} & \text{for } d = 1 \\ \frac{R}{dm} + \frac{1}{m(m-1)} & \text{for } d = 2, \dots, m/R - 1 \\ \frac{R\ln(R/\delta)}{m} + \frac{1}{m(m-1)} & \text{for } d = m/R \\ \frac{1}{m(m-1)} & \text{for } d = m/R + 1, \dots, m \end{cases} \tag{10}$$

where $R = c\log(m/\delta)\sqrt{m}$ for some $c > 0$ and $\delta \in [0, 1]$, with c and δ being design parameters. Some guidelines for choosing c and δ can be found in [21]. The probability of choosing $d = d_0$ is equal to $\rho(d_0)/\sum_{i=1}^{m}\rho(i)$. Once the degree d is chosen, encoding is performed by choosing d source symbols uniformly at random (this determines $\mathcal{S}_d$) and adding them to generate an encoded symbol. The encoding process is illustrated in Figure 1a.

Figure 4, shows simulation results for the number of symbols decoded successfully for each encoded symbol received. For this we perform LT-Coded multiplication of a randomly generated $10,000 \times 10,000$ matrix with a $10,000 \times 1$ vector. The matrix $\mathbf{A}$ is encoded using an LT code with parameters c and δ chosen according to the guidelines of [21]. We generate a single row of the encoded matrix $\mathbf{A_e}$ at a time which is then multiplied with the $10,000 \times 1$ size vector $\mathbf{x}$ to give a single element of the encoded matrix vector product $\mathbf{b_e}$. The process is repeated until we have enough symbols for successfully decoding the entire $10,000 \times 1$ size vector $\mathbf{b}$ using the peeling decoder. The plots of Figure 4 correspond to different choices of c and δ. In each case we observe an avalanche behavior wherein very few symbols are decoded up to a point (approximately upto $10,000$ encoded symbols received) after which the decoding proceeds very rapidly to completion. This effectively illustrates the fact that the computation overhead of the proposed LT coded matrix vector multiplication strategy is very small ($m_d = m(1 + \epsilon)$).

The theoretical encoding and decoding properties of LT codes are summarized in the following lemmas:

Lemma 2 (Theorem 13 in [12]). *For any constant $\delta > 0$, the average degree of an encoded symbol is $\mathcal{O}(\log(m/\delta))$ where m is the number of source symbols.*

Corollary 1. *Each encoding symbol can be generated using $\mathcal{O}(\log m)$ symbol operations on average.*

Lemma 3 (Theorem 17 in [12]). *For any constant $\delta > 0$ and for a source block with m source symbols, the LT decoder can recover all the source symbols from a set of $m' = m + \mathcal{O}(\sqrt{m}\log^2(m/\delta))$ with probability at least $1 - \delta$.*

Corollary 2. *The decoding threshold m_d as defined in Definition 3 is given by $m_d = m(1 + \epsilon)$ where $\epsilon \to 0$ as $m \to \infty$*

Corollary 3. *Since the average degree of an encoded symbol is $\mathcal{O}(\log(m/\delta))$ the decoding requires $\mathcal{O}(m \log m)$ symbol operations on average.*

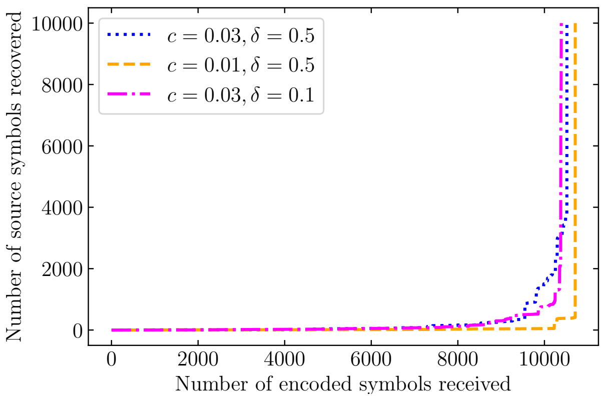

Fig. 4: The number of decoded symbols is almost constant until $m = 10,000$ encoded symbols are received after which it increases rapidly.

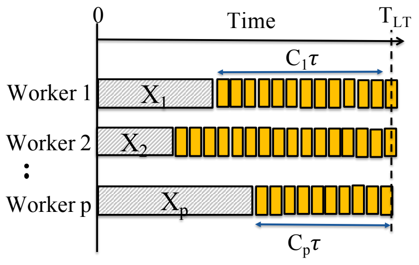

Fig. 5: Worker i has a random exponential initial delay X_i, after which it completes row-vector product tasks (denoted by the small rectangles), taking time τ per task. The latency T_{LT} is the time to complete m_d tasks in total.

B. PROOF OF THEORETICAL RESULTS

B.1. Delay Model and Order Statistics Primer

We first state some standard results [22] on order statistics of exponential random variables to aid the understanding of the latency analysis presented subsequently. If $X_1, X_2, \dots X_p$ are exponential random variables with rate μ, their k^{th} order statistic is denoted by $X_{k:p}$. Thus, $X_{1:p} = \min(X_1, X_2, \dots X_p)$, and $X_{p:p} = \max(X_1, X_2, \dots X_p)$. The expected value of $X_{k:p}$ is given by

$$\mathbb{E}[X_{k:p}] = \frac{1}{\mu}\left(\frac{1}{p} + \cdots + \frac{1}{p-k+1}\right) \tag{11}$$

$$= \frac{H_p - H_{p-k}}{\mu}, \tag{12}$$

where H_p is the p^{th} Harmonic number

$$H_p \triangleq \begin{cases} \sum_{i=1}^{p} \frac{1}{i} & \text{for } p = 1, 2, \dots \\ 0 & \text{for } p = 0 \end{cases} \tag{13}$$

For large p, $H_p = \log p + \gamma$, where γ is the Euler-Mascheroni constant and thus we can use the approximation $H_p \simeq \log p$ for large p.

Also the difference of consecutive order statistics of i.i.d exponential random variables is also exponentially distributed as,

$$(X_{l+1:p} - X_{l:p}) \overset{d}{=} U_{p-l} \tag{14}$$

where $U_{p-l} \sim \exp((p-l)\mu)$.

B.2. Proof of Theorem 1

As per our model, the time taken by worker i to perform B_i computations is given by

$$Y_i = X_i + \tau B_i, \text{ for } i = 1, \dots, p. \tag{15}$$

The latency T_{LT} is the earliest time when $\sum_{i=1}^{p} B_i = m_d$, as illustrated in Figure 2. We note that, in this case it is not necessary that each worker has completed at least 1 computation. Specifically, if $T_{\text{LT}} - X_i \leq \tau$ for any i then it means that worker i has not performed even a single computation in the time that the system as a whole has completed m_d computations (owing to the large initial delay X_i). Therefore we define

$$\mathcal{W}_{\text{LT}} := \{i : T_{\text{LT}} - X_i \geq \tau\} \tag{16}$$

Here $\mathcal{W}_{\text{LT}}$ is the set of workers for which $B_i > 0$. Thus

$$\begin{aligned} T_{\text{LT}} &= \max_{i \in \mathcal{W}_{\text{LT}}} Y_i, && (17)\\ &= \max_{i \in \mathcal{W}_{\text{LT}}} \left(X_i + \tau B_i\right), && (18)\\ &\geq \min_{i \in \{1, \dots p\}} X_i + \tau \max_{i \in \mathcal{W}_{\text{LT}}} B_i, && (19)\\ &\geq \min_{i \in \{1, \dots p\}} X_i + \tau \frac{m'}{p}, && (20)\end{aligned}$$

where to obtain (19), we replace each X_i in (18) by $\min_{i \in [1, \dots p]} X_i$ and then we can bring it outside the maximum. To obtain (20), we observe that in order for the p workers to collectively finish m' computations, the maximum number of computations completed by a worker has to be at least m/p. Taking expectation on both sides we get

$$\begin{aligned} \mathbb{E}[T_{\text{LT}}] &\geq \mathbb{E}[\min\left(X_1, X_2, \dots X_p\right)] + \frac{\tau m'}{p}, && (21)\\ &= \frac{1}{p\mu} + \frac{\tau m'}{p}. && (22)\end{aligned}$$

where the lower bound in (22) follows from the result (12) on order statistics of exponential random variables. To derive the upper bound, we note that

$$T_{\text{LT}} \leq X_i + \tau(B_i + 1), \quad \text{for all } i = 1, \dots, p \tag{23}$$

This is because at time T_{LT} each of the workers $1, \dots, p$, have completed $B_1, \dots, B_p$ row-vector product tasks respectively, but they may have partially completed the next task. The 1 added to each B_i accounts for this edge effect, which is also illustrated in Figure 2. Summing over all i on both sides, we get

$$\begin{aligned} \sum_{i=1}^{p} T_{\text{LT}} &\leq \sum_{i=1}^{p} X_i + \sum_{i=1}^{p} \tau\left(B_i + 1\right) && (24)\\ pT_{\text{LT}} &\leq \sum_{i=1}^{p} X_i + \tau\left(m_d + p\right) && (25)\end{aligned}$$

Taking expectation on both sides and rearranging we obtain the upper bound,

$$\begin{aligned} p\mathbb{E}[T_{\text{LT}}] &\leq \frac{p}{\mu} + \tau\left(m_d + p\right), && (26)\\ \mathbb{E}[T_{\text{LT}}] &\leq \frac{\tau m_d}{p} + \frac{1}{\mu} + \tau. && (27)\end{aligned}$$

B.3. Proof of Theorem 2

In the $r-$replication strategy each submatrix $\mathbf{A}_i$ is replicated at r workers and we wait for the fastest of these r workers. Without loss of generality, we assume that submatrix $\mathbf{A}_1$ is stored at workers $1, 2, \dots, r$, submatrix $\mathbf{A}_2$ is stored at workers $r+1, r+2, \dots, 2 * r$ and so on. More generally submatrix $\mathbf{A}_i$ is stored at workers $(i-1)r + 1, \dots, ir$. Thus the time taken to compute the product $\mathbf{A}_i\mathbf{x}$ is given by

$$\begin{aligned} Z_i &= \min\left(Y_{(i-1)r+1}, Y_{(i-1)r+2}, \dots, Y_{ir}\right) && (28)\\ &= \min\left(X_{(i-1)r+1} + \tau B_{(i-1)r+1}, \dots, X_{ir} + \tau B_{ir}\right) && (29)\\ &= \min\left(X_{(i-1)r+1}, \dots, X_{ir}\right) + \frac{\tau m r}{p} && (30)\\ &= W_i + \frac{\tau m r}{p} && (31)\end{aligned}$$

where $W_i = \min(X_{(i-1)r+1}, \dots, X_{ir})$ is an $\exp(r\mu)$ random variable since it is the minimum of r $\exp(\mu)$ random variables. This is because the fastest of the r workers that store $\mathbf{A}_i$ is the one corresponding to $\min(X_{(i-1)r+1}, \dots, X_{ir})$ and this worker must perform $\frac{mr}{p}$ computations to compute the product $\mathbf{A}_i\mathbf{x}$.

The latency T_{rep} is the time at which the product $\mathbf{A}_i\mathbf{x}$ is computed for all $i = 1, \ldots, p/r$ since $\mathbf{A}$ is split into p/r submatrices. Thus

$$\begin{aligned} T_{\text{rep}} &= \max\left(Z_1, Z_2, \ldots, Z_{p/r}\right), && (32)\\ &= \max\left(W_1, W_2, \ldots, W_{p/r}\right) + \frac{\tau m r}{p}, && (33) \end{aligned}$$

Taking expectation on both sides

$$\begin{aligned} \mathbb{E}[T_{\text{rep}}] &= \frac{\tau m r}{p} + \mathbb{E}[\max\left(W_1, W_2, \ldots, W_{p/r}\right)], && (34)\\ &= \frac{\tau m r}{p} + \frac{1}{r\mu} H_{p/r}, && (35)\\ &\simeq \frac{\tau m r}{p} + \frac{1}{r\mu}\log\frac{p}{r}, && (36) \end{aligned}$$

where (35) and (36) follow from (12) and (13). This proves the result for the expected latency of the replication strategy.

The latency in the MDS-coded case is $T_{\text{MDS}} = Y_{k:p}$, where $Y_{k:p}$ is the k^{th} order statistic of the individual worker latencies $Y_1, Y_2, \ldots, Y_p$ since we only wait for the fastest k workers to finish the task assigned to them. In this case, each of the fastest k workers performs $\frac{m}{k}$ computations and thus the expected overall latency is given by

$$\begin{aligned} \mathbb{E}[T_{\text{MDS}}] &= \mathbb{E}[X_{k:p}] + \tau\frac{m}{k}, && (37)\\ &= \frac{\tau m}{k} + \frac{1}{\mu}\left(H_p - H_{p-k}\right), && (38)\\ &\simeq \frac{\tau m}{k} + \frac{1}{\mu}\log\frac{p}{p-k}. && (39) \end{aligned}$$

where (38) and (39) follow from the exponential order statistics results in (12) and (13).

B.4. Proof of Theorem 3

As per our model, we represent the number of computations at worker i by the random variable B_i . We also use the random variable C_{MDS} to denote the total number of computations performed by all p workers until T_{MDS}, which is the time when the master collects enough computations to be able to recover the matrix-vector product $\mathbf{b} = \mathbf{A}\mathbf{x}$. Thus

$$\begin{aligned} C_{\text{MDS}} &= B_1 + B_2 + \ldots + B_p && (40)\\ &= B_{1:p} + B_{2:p} + \ldots + B_{p:p}, && (41) \end{aligned}$$

where the second expression is simply the right-hand side of the first expression written in terms of the corresponding order statistics. We note that under our model the time spent by worker i in performing B_i computations is $Y_i = X_i + \tau B_i$ where X_i denotes setup/initial delay and τ is a constant denoting the time taken to perform a single computation. Thus $B_{1:p}$ corresponds to the worker that performs the least number of computations which is also the worker with the largest value of setup time i.e $X_{p:p}$ since all workers stop computing at the same time (T_{MDS}). Thus for a given C_0, the tail of the total number of computations performed in the MDS Coded strategy is given by

$$\begin{aligned} \Pr\left(C_{\text{MDS}} \le \frac{mp}{k} - C_0\right) &= \Pr\left(\sum_{i=1}^{p} B_{i:p} \le \frac{mp}{k} - C_0\right) && (42)\\ &= \Pr\left(\sum_{i=1}^{p-k} B_{i:p} + \frac{m}{k} \times k \le \frac{mp}{k} - C_0\right) && (43)\\ &= \Pr\left(\sum_{i=1}^{p-k} B_{i:p} \le \frac{m\left(p-k\right)}{k} - C_0\right) && (44)\\ &\le \Pr\left(\left(p-k\right) B_{1:p} \le \frac{m\left(p-k\right)}{k} - C_0\right) && (45)\\ &= \Pr\left(B_{1:p} \le \frac{m}{k} - \frac{C_0}{p-k}\right) && (46) \end{aligned}$$

where (43) follows from the fact that the fastest k workers which correspond to $B_{p-k+1:p}, B_{p-k+2:p}, \ldots, B_{p:p}$ must perform all the tasks assigned to them i.e. m/k computations each, while (45) follows from the fact that $B_{2:p}, \ldots, B_{p:p}$ are always larger than $B_{1:p}$ by definition.

At this point we note that the worker which performs $B_{1:p}$ computations has setup time $X_{p:p}$. There can be two possibilities – either $T_{\text{MDS}} > X_{p:p}$, or $T_{\text{MDS}} \leq X_{p:p}$. If $T_{\text{MDS}} > X_{p:p}$ then

$$T_{\text{MDS}} \leq X_{p:p} + \tau \left(B_{1:p} + 1\right) \tag{47}$$

where the added 1 accounts for the edge effect of partial computations at the nodes. If $T_{\text{MDS}} \leq X_{p:p}$ then also the upper bound (47) holds. Thus overall (by rearranging terms in (47)) we obtain,

$$B_{1:p} \geq \frac{T_{\text{MDS}} - X_{p:p}}{\tau} - 1 \tag{48}$$

Thus we can write

$$\begin{aligned} \Pr\left(C_{\text{MDS}} \leq \frac{mp}{k} - C_0\right) &\leq \Pr\left(\frac{T_{\text{MDS}} - X_{p:p}}{\tau} - 1 \leq \frac{m}{k} - \frac{C_0}{p-k}\right) && (49)\\ &= \Pr\left(X_{p:p} - X_{k:p} \geq \frac{\tau C_0}{p-k} - \tau\right) && (50)\\ &= \Pr\left(\sum_{l=k}^{p-1}\left(X_{l+1:p} - X_{l:p}\right) \geq \frac{\tau C_0}{p-k} - \tau\right) && (51)\end{aligned}$$

where (50) follows from the fact that $T_{\text{MDS}} = X_{k:p} + \tau m/k$.

If $X_i \sim \exp(\mu)$ we can use the result from (14) to simplify the above expression further,

$$\begin{aligned} \Pr\left(C_{\text{MDS}} \leq \frac{mp}{k} - C_0\right) &\leq \Pr\left(\sum_{i=1}^{p-k} U_i \geq \frac{\tau C_0}{p-k} - \tau\right) && (52)\\ &\leq \Pr\left(\left(p-k\right)U_1 \geq \frac{\tau C_0}{p-k} - \tau\right) && (53)\\ &= \Pr\left(U_1 \geq \frac{\tau C_0}{\left(p-k\right)^2} - \frac{\tau}{p-k}\right) && (54)\\ &= \exp\left(-\mu\left(\frac{\tau C_0}{\left(p-k\right)^2} - \frac{\tau}{p-k}\right)\right) && (55)\end{aligned}$$

where (53) is obtained from the fact that $\Pr(U_1 \geq u) \geq \Pr(U_l \geq u)$ for $l = 2, \ldots, p-k$ for any u since $U_l \sim \exp(l\mu)$. Lastly (55) is obtained from the expression for the tail distribution of an exponential random variable.

B.5. Proof of Theorem 4

As per our model, we represent the number of computations at worker i by the random variable B_i . We also use the random variable C_{rep} to denote the total number of computations performed by all p workers until T_{rep}, which is the time when the master collects enough computations to be able to recover the matrix-vector product $\mathbf{b} = \mathbf{A}\mathbf{x}$. Thus

$$\begin{aligned} C_{\text{rep}} &= B_1 + B_2 + \ldots + B_p && (56)\\ &= \sum_{i=1}^{p/r}\sum_{j=1}^{r} B_{(i-1)r+j}, && (57)\end{aligned}$$

where the term inside the summation in the second expression represents the number of computations performed by each worker that store a copy of the submatrix $\mathbf{A}_i$ (for a given i). In what follows, we use the shorthand notation $D_j^i = B_{(i-1)r+j}$ and use $D_{j:r}^i$ to denote the order statistics of $D_1^i, \ldots, D_r^i$. Rewriting the above expression in terms of the order statistics we get,

$$C_{\text{rep}} = \sum_{i=1}^{p/r}\sum_{j=1}^{r} D_{j:r}^i, \tag{58}$$

and the tail bound,

$$\Pr(C_{\text{rep}} \leq mr - C_0) = \Pr\left(\sum_{i=1}^{p/r}\sum_{j=1}^{r} D^i_{j:r} \leq mr - C_0\right) \tag{59}$$

$$= \Pr\left(\sum_{i=1}^{p/r}\sum_{j=1}^{r-1} D^i_{j:r} \leq m(r-1) - C_0\right) \tag{60}$$

$$\leq \Pr\left((r-1)\sum_{i=1}^{p/r} D^i_{1:r} \leq m(r-1) - C_0\right) \tag{61}$$

$$= \Pr\left(\sum_{i=1}^{p/r} D^i_{1:r} \leq m - \frac{C_0}{r-1}\right) \tag{62}$$

where (60) follows from the fact that for any given submatrix $\mathbf{A}_i, i = 1, \dots, p/r$, the fastest worker that stores a copy of that submatrix, which corresponds to $D^i_{r:r}$ (fastest worker performs the most computations) must perform all the tasks assigned to it i.e. mr/p computations each, while (61) follows from the fact that $D^i_{2:r}, \dots, D^i_{r:r}$ are always larger than $D^i_{1:r}$ by definition.

At this point we introduce the shorthand notation $V^i_j = X_{(i-1)r+j}$ for the setup time of the worker that stores the j^{th} copy of submatrix $\mathbf{A}_i$ and note that the worker which performs $D^i_{1:r}$ computations has setup time $V^i_{r:r}$ ($V^i_{j:r}$ are the order statistics of $V^i_1, \dots, V^i_r$). There can be two possibilities – either $T_{\text{rep}} > V^i_{r:r}$, or $T_{\text{rep}} \leq V^i_{r:r}$. If $T_{\text{rep}} > V^i_{r:r}$ then

$$T_{\text{rep}} \leq V^i_{r:r} + \tau(D^i_{1:r} + 1) \tag{63}$$

where the added 1 accounts for the edge effect of partial computations at the nodes. If $T_{\text{rep}} \leq V^i_{r:r}$ then also the upper bound (63) holds. Thus overall (by rearranging terms in (63)) we obtain,

$$D^i_{1:r} \geq \frac{T_{\text{rep}} - V^i_{r:r}}{\tau} - 1 \tag{64}$$

Thus we can write

$$\Pr(C_{\text{rep}} \leq mr - C_0) \leq \Pr\left(\sum_{i=1}^{p/r}\left(\frac{T_{\text{rep}} - V^i_{r:r}}{\tau} - 1\right) \leq m - \frac{C_0}{r-1}\right) \tag{65}$$

$$= \Pr\left(\sum_{i=1}^{p/r}(V^i_{r:r} - W_{\text{rep}}) \geq \frac{\tau C_0}{r-1} - \frac{\tau p}{r}\right) \tag{66}$$

where $W_{\text{rep}} = \max_i V^i_{1:r} = \max_{1\leq i\leq p/r} \min_{1\leq j\leq r} X_{(i-1)r+j}$ and (66) follows from the fact that

$$T_{\text{rep}} = \max_{1\leq i\leq p/r} \min_{1\leq j\leq r} X_{(i-1)r+j} + \tau mr/p$$

From our definition of W_{rep} we see that,

$$V^i_{r:r} - W_{\text{rep}} \leq V^i_{r:r} - V^i_{1:r} \tag{67}$$

and the consequent stochastic dominance can be used to get an upper bound on (66) as,

$$\Pr(C_{\text{rep}} \leq mr - C_0) \leq \Pr\left(\sum_{i=1}^{p/r}(V^i_{r:r} - V^i_{1:r}) \geq \frac{\tau C_0}{r-1} - \frac{\tau p}{r}\right) \tag{68}$$

$$= \Pr\left(\sum_{i=1}^{p/r}\sum_{j=1}^{r-1}(V^i_{j+1:r} - V^i_{j:r}) \geq \frac{\tau C_0}{r-1} - \frac{\tau p}{r}\right) \tag{69}$$

If $X_i \sim \exp(\mu)$ we can use the result from (14) to simplify the above expression further (since $V_j^i = X_{(i-1)r+j}$ are also exponentially distributed and thus $(V_{j+1:r}^i - V_{j:r}^i) \overset{d}{=} U_{r-j}^i$, $U_{r-j}^i \sim \exp((r-j)\mu)$),

$$\Pr(C_{\text{rep}} \leq mr - C_0) \leq \Pr\left(\sum_{i=1}^{p/r}\sum_{j=1}^{r-1} U_{r-j}^1 \geq \frac{\tau C_0}{r-1} - \frac{\tau p}{r}\right) \tag{70}$$

$$\leq \Pr\left((r-1)\sum_{i=1}^{p/r} U_1^i \geq \frac{\tau C_0}{r-1} - \frac{\tau p}{r}\right) \tag{71}$$

$$= \Pr\left(\sum_{i=1}^{p/r} U_1^i \geq \frac{\tau C_0}{(r-1)^2} - \frac{\tau p}{r(r-1)}\right) \tag{72}$$

$$= \sum_{i=0}^{p/r-1} \frac{1}{i!}\exp(-\mu\theta)(\mu\theta)^i. \tag{73}$$

where (71) is obtained from the fact that $\Pr(U_1^i \geq u) \geq \Pr(U_{r-j}^i \geq u)$ for $j = 1, \ldots, r-2$ for any u since $U_{r-j} \sim \exp((r-j)\mu)$. Lastly (73) is obtained from the expression for the tail distribution of an Erlang random variable which is the sum of p/r exponential random variables with rate μ and

$$\theta = \frac{\tau C_0}{(r-1)^2} - \frac{\tau p}{r(r-1)} \tag{74}$$